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ELASTIC MINIMUM-WEIGHT DESIGN FOR SPECIFIED FUNDAMENTAL FREQUENCY*

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Abstract—A one-dimensional structure with segmentwise constant specific stiffness is to have a prescribed fundamental frequency. The number of segments of constant specific stiffness is given, but their boundaries and specific stiffnesses are to be determined to minimize the structural weight. A sufficient condition for optimality is established and its use is illustrated by an example.

1. THEORY

THE problem of minimum mass design with specified natural frequencies has been investigated by Turner [1]. In his analysis, the continuous structure was replaced by a system with a finite number of degrees of freedom, and the problem was formulated as a Lagrange problem in variational calculus with the free vibration equations as side conditions. Taylor [2] discussed the same problem using an energy approach. In the present paper, the following more realistic problem is considered: a one-dimensional† elastic structure (rod, beam, frame) with segmentwise constant specific‡ stiffness is to carry given point masses at specified locations in addition to its own mass and to have a prescribed fundamental frequency ω. The number n of segments of constant specific stiffness is prescribed, but their boundaries and specific stiffnesses are at the choice of the designer who wishes to minimize the structural weight.

The location of a cross section of the one-dimensional structure will be specified by its distance x (measured along the structure) from a fixed reference section. If it is desirable to indicate that a cross section is in the ith segment of constant stiffness, its location will be referred to as x_i rather than x. The location of the point mass M_α will be denoted by x_α^*, $(\alpha = 1, 2, \ldots, \nu)$. The specific mass of the structure will be assumed to be a linear function of the specific stiffness; in particular, the specific mass of the ith segment will be written as $a_i^2 + b_i^2 s_i$, where s_i is the specific stiffness of the ith segment and a_i and b_i are given constants.

The assumption of a linear relation between specific mass and stiffness does impose some limitations on the type of the structural members. However, from the point of view that every continuous function can be well approximated by successive linear functions, the limitation is not too severe, and the assumption will simplify the solution of the problem.

In practical design, the constants a_i, b_i for an element are known only after the approximate value of the stiffness of the segment is known. In this case, some approximate values

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† For brevity, the theory is developed for one-dimensional structures, but its extension to two-dimensional structures (disks, plates, shells) presents no difficulties.

‡ Throughout this paper, the term "specific" will be used to indicate "per unit length."

of a_i and b_i, for example an average value for the sections available for the design, can be used in a first solution of the problem which furnishes first values of the stiffnesses on the basis of which better values may be chosen for the a_i and b_i. This kind of iteration can be continued, if necessary.

As a first step toward the solution of the problem, assume that the boundaries of the segments of constant stiffness are given so that only the stiffness values s_i remain to be determined. An ordered set of n stiffness values then specifies a design. Let s_i and $\bar{s}_i$, $(i = 1, 2, \ldots, n)$ be two designs with the same fundamental frequency ω, and denote their fundamental modes by $u(x)$ and $\bar{u}(x)$. From the fact that the square of the fundamental frequency can be written as a Rayleigh quotient, there follows the relation

$$\omega^2 = \frac{\sum_i s_i \int e[u(x_i)]\,\mathrm{d}x_i}{\sum_i (a_i^2 + b_i^2 s_i) \int u^2(x_i)\,\mathrm{d}x_i + \sum_\alpha M_\alpha u^2(x_\alpha^*)} = \frac{\sum_i \bar{s}_i \int e[\bar{u}(x_i)]\,\mathrm{d}x_i}{\sum_i (a_i^2 + b_i^2 \bar{s}_i) \int \bar{u}^2(x_i)\,\mathrm{d}x_i + \sum_\alpha M_\alpha \bar{u}^2(x_\alpha^*)}, \tag{1}$$

where $s_i e[u(x_i)]$ is twice the specific strain energy associated with the displacement $u(x_i)$. For example, if the ith segment of a rod has the cross section A_i, then $s_i e[u_i(x_i)] = EA_i[u'(x_i)]^2$, for the axial vibration, where E is Young's modulus and the prime denotes differentiation with respect to x_i. The integrals in (1) are extended over the segment x_i.

Since the displacement field $u(x)$ is kinematically admissible for the design $\bar{s}_i$, $(i = 1, 2, \ldots, n)$, it follows from Rayleigh's principle that

$$\omega^2 = \frac{\sum_i \bar{s}_i \int e[\bar{u}(x_i)]\,\mathrm{d}x_i}{\sum_i (a_i^2 + b_i^2 \bar{s}_i) \int \bar{u}^2(x_i)\,\mathrm{d}x_i + \sum_\alpha M_\alpha \bar{u}^2(x_\alpha^*)} \le \frac{\sum_i \bar{s}_i \int e[u(x_i)]\,\mathrm{d}x_i}{\sum_i (a_i^2 + b_i^2 \bar{s}_i) \int u^2(x_i)\,\mathrm{d}x_i + \sum_\alpha M_\alpha u^2(x_\alpha^*)}. \tag{2}$$

The equality involving the first and second members in (1) and the inequality involving the first and third members in (2) furnish the relations

$$\sum_i s_i \int \{e[u(x_i)] - b_i^2 \omega^2 u^2(x_i)\}\,\mathrm{d}x_i = \omega^2 \{\sum_i a_i^2 \int u^2(x_i)\,\mathrm{d}x_i + \sum_\alpha M_\alpha u^2(x_\alpha^*)\}, \tag{3}$$

$$\sum_i \bar{s}_i \int \{e[u(x_i)] - b_i^2 \omega^2 u^2(x_i)\}\,\mathrm{d}x_i \ge \omega^2 \{\sum_i a_i^2 \int u^2(x_i)\,\mathrm{d}x_i + \sum_\alpha M_\alpha u^2(x_\alpha^*)\}. \tag{4}$$

Subtraction of (3) from (4) yields

$$\sum_i (\bar{s}_i - s_i) \int \{e[u(x_i)] - b_i^2 \omega^2 u^2(x_i)\}\,\mathrm{d}x_i \ge 0. \tag{5}$$

Now, the difference between the structural weights of the designs $\bar{s}_i$ and s_i is given by

$$\Delta W = \sum_i b_i^2 (\bar{s}_i - s_i) l_i, \tag{6}$$

where l_i is the length of the ith segment. Inspection of (5) and (6) shows that ΔW will be nonnegative if

$$C = \frac{1}{b_i^2 l_i} \int \{e[u(x_i)] - b_i^2 \omega^2 u^2(x_i)\} \, dx_i \quad \text{is independent of } i. \tag{7}$$

Accordingly, a design with the prescribed frequency ω and a fundamental mode $u(x)$ satisfying (7) will not be heavier than any other design with the same fundamental frequency.

The optimality condition (7) has been shown to be sufficient. That it is also necessary can be established in the manner used in [3] to prove the necessity of a similar optimality condition for elastic minimum-weight design for prescribed stiffness.

As the length l_i of the typical segment tends to zero, the optimality condition (7) tends to the condition given by Prager and Taylor [4] for structures with continuously varying specific stiffness where b_i is a constant. This condition requires the integrand in (7) divided by b_i^2 to be constant along the structure.

Theoretically, the problem can be solved in the following procedures: (a) From the equations of motion and the boundary (or transition) conditions for all segments, solve for the displacements u_i in terms of the fundamental frequency ω, the stiffnessses s_i, and the lengths l_i of the segments. (b) Determine the stiffnesses s_i, in terms of ω and the lengths l_i, from the optimality conditions (7) and the boundary (or transition) conditions involving the point masses. (c) The total weight of the structure is minimized with respect to the ratios of the lengths l_i. A simple problem is treated in the following example.

2. EXAMPLE

A vertical rod is fixed at the upper end $x = 0$ and carries a mass M at the lower end $x = l$; it is to have the given fundamental frequency ω in longitudinal vibration. Each of the segments $0 \leq x < l_1$ and $l_1 < x \leq l = l_1 + l_2$ of the rod is to have constant specific stiffness. The stiffness values s_1 and s_2 are to be determined to minimize the weight of the rod.

Since specific mass and stiffness of a prismatic rod are ρA and EA, where ρ and E respectively denote density and Young's modulus of the rod material, $a_i^2 = 0$ and $b_i^2 = \rho/E = b^2$. Moreover, $e = [u'(x)]^2$. With the abbreviations u_1 and u_2 for $u(x_1)$ and $u(x_2)$, the optimality condition (7) becomes

$$\frac{1}{l_1} \int_0^{l_1} (u_1'^2 - b^2 \omega^2 u_1^2) \, dx_1 = \frac{1}{l_2} \int_{l_1}^{l} (u_2'^2 - b^2 \omega^2 u_2^2) \, dx_2. \tag{8}$$

The equations of motion are

$$u_1'' + b^2 \omega^2 u_1 = 0, \qquad u_2'' + b^2 \omega^2 u_2 = 0, \tag{9}$$

with the boundary conditions

$$u_1(0) = 0, \qquad EA_2 u_2'(l) = M\omega^2 u_2(l), \tag{10}$$

and the transition conditions

$$u_1(l_1) = u_2(l_1), \qquad A_1 u_1'(l_1) = A_2 u_2'(l_1). \tag{11}$$

The problem (9), (10), (11) is homogeneous. In view of the first boundary condition (10) and the transition conditions (11), the fundamental mode $u(x)$ has the form

$$\begin{aligned} u_1 &= cA_2 \sin b\omega x, \\ u_2 &= c[A_1 \cos b\omega l_1 \sin b\omega(x-l_1) + A_2 \sin b\omega l_1 \cos b\omega(x-l_1)], \end{aligned} \tag{12}$$

where c is an arbitrary constant with dimension $(\text{length})^{-1}$.

Equations for the cross-sectional areas A_1 and A_2 may now be obtained by substituting (12) into the optimality condition (8) and the second boundary condition (10). With $\eta = A_1/A_2$, the optimality condition furnishes the quadratic equation

$$\eta^2 \cos^2 b\omega l_1 \sin 2b\omega l_2 - 2\eta \sin^2 b\omega l_2 \sin 2b\omega l_1 - \sin^2 b\omega l_1 \sin 2b\omega l_2 - \frac{l_2}{l_1} \sin 2b\omega l_1 = 0. \tag{13}$$

Note that η is positive by definition. The domain $\eta > 0$ is shown in Fig. 1. Note that the

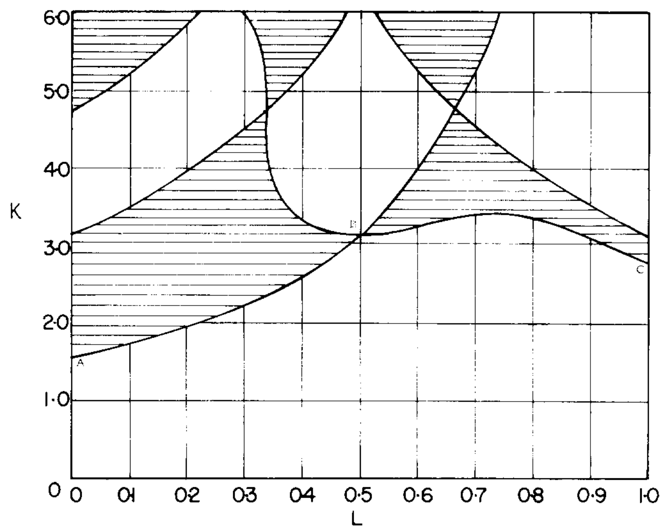

FIG. 1. Equation (13) has precisely one positive root in open unshaded region and no positive root in closed shaded region.

curve ABC represents the least upper bound of K of the fundamental frequency. If (13) has a positive root η for the given values of $K = b\omega l$ and $L = l_1/l$, the cross-sectional areas of the optimal design are obtained from the second boundary condition (10). Doing this, one finds

$$\begin{aligned} A_2 &= \frac{\sin KL \cos K(1-L) + \eta \cos KL \sin K(1-L)}{\eta \cos KL \cos K(1-L) - \sin KL \sin K(1-L)} \left(\frac{\omega^2 Ml}{KE}\right) \\ A_1 &= \eta A_2. \end{aligned} \tag{14}$$

The weight of the optimal design for given L is proportional to the dimensionless volume

$$V = \{1-(1-\eta)L\} \frac{KEA_2}{\omega^2 Ml} \tag{15}$$

where the dependence of A_1, A_2 and η on L and K is expressed by (14) and (13). If L is a variable at the choice of the designer rather than being prescribed, the expression (15) must be minimized with respect to L under the constraints (14) and (13). Over an essential range of K, the curves marked L and η in Fig. 2 show the dependence of the values of L and η for the optimal design on K. Similarly, the curves marked V and R in Fig. 3 show the

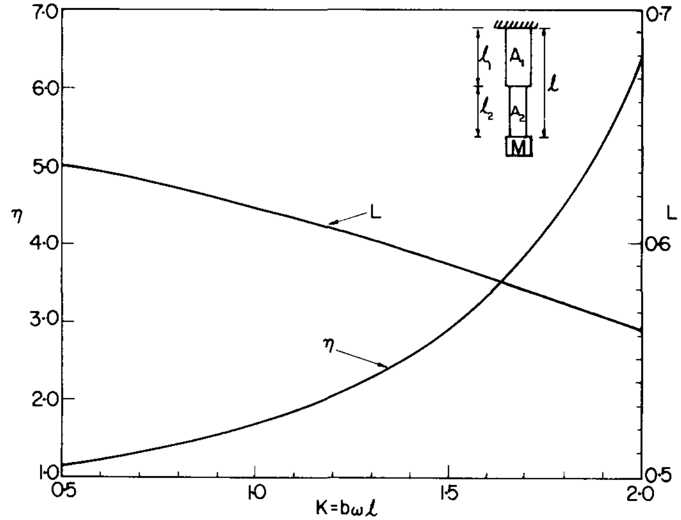

FIG. 2. L and $\eta = A_1/A_2$ vs. $K = b\omega l$.

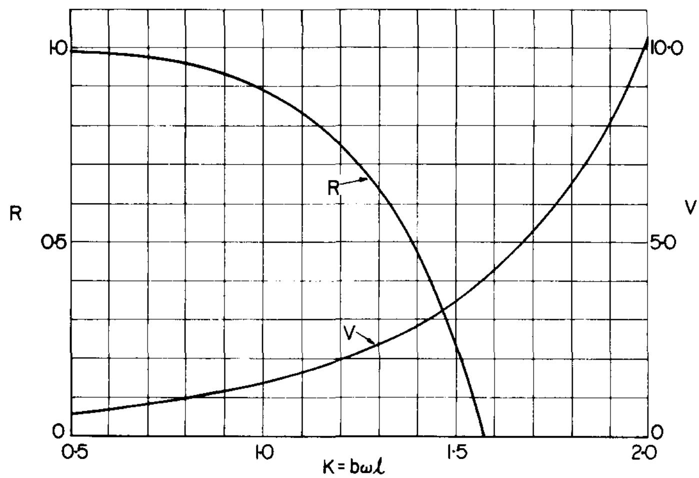

FIG. 3. R (ratio of volumes of optimal design and rod of constant cross section) and V (dimensionless volume of optimal design) vs. $K = b\omega l$.

variations of the dimensionless volume V of the optimal design and the ratio R of this volume to the volume of the prismatic rod with the same fundamental frequency:

$$R = \frac{A_1 l_1 + A_2(l - l_1)}{(\omega^2 M l^2/EK)\tan K}. \tag{16}$$

Note that $R = 0$ for $K = \pi/2$, which is the maximum of K for a rod of constant stiffness. The fact that R is close to 1 for $K < 0.5$ implies that, for low frequencies, the optimal design is nearly prismatic.

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Абстракт—Одномерная конструкция, с постоянной специфической жесткостью отдельных сегментов, обладает заданной основной частотой. Число сегментов для постоянной специфической жескости задано, но их границы и специфические жесткости следует определить путем минимализации веса конструкции. Устанавливается достаточное чсловие оптимальности. Ее использование иллюстрируется примером.